Note: Guidelines were changed from 5000 characters to 2-pages

STAR Research Statement: Computational Geometry and the All-Net Property of Polytopes

[redacted]

Background and Context

The exploration of unfolding polyhedra gained prominence thanks to Albrecht Dü ̈rer's work in the early 1500s, notably his significant publication The Painter's Manual [3]. Within this text lies the initial documented instance, wherein a polyhedron is sliced along select edges and unfolded to a single, non-overlapping, simple polygon in the plane. Such an unfolding is called a net. Inspired by this, mathematicians proposed the conjecture that for any convex polyhedron there exists a way to cut along the edges such that the resulting unfolding is a net [5]. However, randomly cutting edges might not necessarily give a net (a non-overlapping unfolding) [2]. Unfolding polyhedra is one of the biggest topics in computational geometry, yet there is little research being done on the "all-net property".

It has been observed that certain polyhedra exemplify an even more specific property. Take for example, the cube. No matter how one cuts along the edges of the cube, it will admit a net (i.e. a connected, non-overlapping unfolding). This attribute has been coined the "all-net property" [2]. Our research aims to establish which conditions must be present to ensure the "all-net property" of a polytope and which conditions will always produce the "all-net property." We refer to these classifications of conditions as "necessary" and "sufficient." Although there continues to be active research in the topic of unfolding polyhedra, there is a lack of research being done on the "all-net property."

Furthermore, this concept can also be extended to higher dimensions, moving from polyhedra to poly- topes. For example, it was shown in 2017 by SURE students at USD that every unfolding of any dimensional cube will be a non-intersecting net. For instance, they showed that of the 9694 distinct unfoldings of 5D cubes into 4D, all are nets (none of the faces overlap) [1]. This is one example of the all-net property, but our research could explain what makes this possible. Driven by the broader goal of contributing to computational geometry, our project builds upon prior research by Dr. [redacted], particularly in the realm of unfolding.

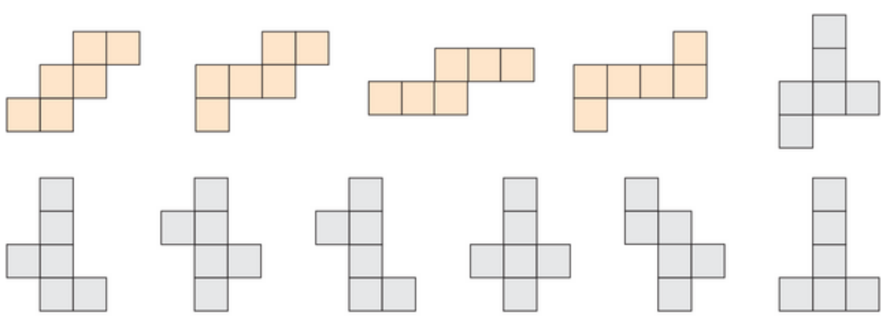

Figure 1: The 11 distinct unfoldings of the cube. The cube possesses the all-net property.

Deliverables / Objectives

The main goal is to identify and articulate necessary and sufficient conditions for the all-net property of polytopes. Research will begin with existing literature on unfoldings, including previous research by Dr. [redacted]

The first third of our research will be focused on 3D polyhedra and formulating necessary conditions. Our team will focus on investigating categories of polyhedra, such as convex, spherical, orthogonal, and truncated polyhedra, to see which classifications, if any, could determine the all-net property. After determining necessary conditions, the next third of the summer will focus on sufficient conditions, investigating paths of cutting, also called spanning trees, as a basis for classifying nets. We will work to understand how the combinatorics of spanning trees are related to geometric unfoldings. Building off of results for polyhedra, we can generalize our findings to higher dimensions in the final third, utilizing software to visualize unfoldings. Finally, we will summarize our findings, setting up the groundwork for an eventual paper.

Methods / Activities

Our approach involves investigating potential necessary and sufficient conditions for the all-net property. Additionally, we will consider cases of examples to substantiate our findings. Activities encompass theorem proving, mathematical modeling, and potential computational aspects. Utilizing software such as Sam Zhang's "Ridge Unfolding Polytopes" program will allow us to visualize unfoldings in higher dimensions [6]. Additionally, this problem is well-suited for hands-on work, which can be completed using the Math Studio. 4D unfoldings, for example, can be visualized as 3D models.

Resources to be Used

Funding will go towards Math Studio supplies and possibly books, enhancing our research capabilities. Spe- cific software, tools, and literature may be utilized, with a plan to acquire proficiency as needed. Leveraging the prior research by Dr. [redacted]ides a foundation for our exploration.

Anticipated Outcomes

A successful project is anticipated to establish clear and comprehensive necessary and sufficient conditions for the all-net property of polytopes. Insights into polyhedral classes as potential conditions will contribute to the computational geometry field. Our research will provide the foundation for a future published paper on this topic.

References

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[3] Albrecht Dü¨rer. Underweysung der Messung mit dem Zirckel und Richtscheyt. Nuremberg, 1525. Also known as: Treatise on Measurement with Compass and Ruler.

[4] Joseph O'Rourke. Unfolding polyhedra, 2019.

[5] G. Shephard. Convex polytopes with convex nets. Mathematical Proceedings of the Cambridge Philosophical Society, 78:389–403, 1975.

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