Example SURE Proposal: Math/Computer Science

Purpose:

This proposal seeks to present a research project in Matrix Analysis. Previously, Dr. Donald Yau and Edward C. Kung, in an article "Finding A Matrix of Given Sign Pattern and Line Sums", have done research on matrices of sign patterns with specified line sums. Their article opens up new questions that require further investigation. This proposal discusses our idea to further explore one of the questions which arose as a result of their research.

Objective and Background:

We will attempt to solve an open problem proposed in the Yau and Kung's article. The authors constructed an algorithm that solves the problem P(A,r,c) described as follows: Given an m-by-n sign pattern **A**, and real vectors **r** in $\mathbb{R}^m$ and **c** in $\mathbb{R}^n$, find an m-by-n matrix **B** with sign pattern and row sums **r** and column sums **c**. A sign pattern is an m-by-n matrix whose entries are either **+'s, -'s, or 0's**.

A simple example of this problem is as follows. Suppose that the matrix of sign patterns A and row and column sum vectors, **r** and **c** are given by:

$$A = \begin{bmatrix} + & 0 & - \\ + & - & + \end{bmatrix} \qquad \mathbf{r} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad \mathbf{c} = \begin{bmatrix} 7 \\ -1 \\ -1 \end{bmatrix}.$$

Then the algorithm given in Yau and Kung's paper would produce:

$$B = \begin{bmatrix} 3 & 0 & -2 \\ 4 & -1 & 1 \end{bmatrix}.$$

So B has the sign pattern of A and its column sum is given by **c** and row sum by **r.**

Their paper shows that if a solution exists for the problem P(A,**r**,**c**) referred to above, in which all of the elements of **r** and **c** are integers, then there exists an integer N such that the problem P(A,N**r**,N**c**) also has a solution in which all of its elements are integers. So in the example above if N=3 then we would have,

$$N\mathbf{r} = \begin{bmatrix} 3 \\ 12 \end{bmatrix} \qquad N\mathbf{c} = \begin{bmatrix} 21 \\ -3 \\ -3 \end{bmatrix}.$$

So according to Yau and Kung for some N (perhaps not 3) there would exist a matrix C with integer entries, satisfying the requirement that the row sum of C is N**r** and the column sum of C is N**c.**

Expanding on the above statement, the authors asked "What is the smallest integer N such that the problem P (A, Nr, Nc), where all the elements of **r** and **c** are integers, has an integer solution?" The main goal of this research will be to answer the question raised above, although we will also investigate the question of when the problem P (A, Nr, Nc) has a unique integer solution.

Methodology:

In the article "Possible Line Sums for a Qualitative Matrix," Dr. Donald Yau and Dr. Charles Johnson showed that there is a solution to the problem P (A, r, c) described earlier. However, the authors left the question about how to construct such a solution for further investigation. Yau and Kung's article gives a construction algorithm for finding a solution to P (A, r, c) when it is known to exist. We will study that algorithm and use similar techniques to answer the open questions raised in the article.

Expectations:

We plan to achieve the following during our research in the summer of 2005:

i. Become familiar with the algorithm developed by Dr. Donald Yau and Edward Kung
ii. Determine the smallest integer N such that a problem P (A, Nr, Nc), where all the elements of **r** and **c** are integers, has an integer solution
iii. Determine when the problem P (A, Nr, Nc) has a unique integer solution.
iv. Write up our results and submit them for publication in the Rose-Hullman Undergraduate Math Journal or a similar journal.

Conclusion:

The main goal of this research will be to answer the question of what the smallest integer N such that a problem P (A, Nr, Nc), where all the elements of **r** and **c** are integers, has an integer solution. We will also attempt to answer the question of when the problem P (A, Nr, Nc) has a unique integer solution.

Proofs of these problems will contribute to the efficiency of the algorithm referred to, which can be used to solve optimization problems and also problems in other mathematical oriented subjects such as computer science.

References

Charles R. Johnson, Suzanne A. Lewis, Donald Y. Yau, Possible line sums for a qualitative matrix, Linear Alg. Appl. 327 (2001), no. 1-3, 54-60.

Edward C. Kung, Donald Y. Yau, Finding a matrix of given sign pattern and line sums, Rose-Hullman Undergraduate Math Journal. 5 (2003), no 1.

Charles R. Johnson, D. Stafford, Patterns that allow given row and column sums, Linear Algebra Appl. 311 (2000) 97-105.

R. Brualdi, Convex sets of nonnegative matrices, Can J. Math. 20 (1968) 144-157.

M. Menon, Matrix links, an extrimization problem, and the reduction of a nonnegative matrix to one with prescribed row and column sums, Can J. Math. 20 (1968) 225-232.